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Quantum algorithms for linear systems of equations

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based on joint work
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Outline

- Motivation
- HHL and CKS Algorithms
- New Adiabatic like algorithms
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Motivation

- Quantum computers have the potential to perform computations that are classically intractable.
- Fast quantum algorithms exist for simulating the dynamics of quantum systems and factoring integers.
- Some problems cannot be solved dramatically faster by quantum computers than by classical ones.
- The full power of quantum computers is unknown.
- It is important to find fast quantum algorithms for problems with broad applications.

Problem: System of linear equations

Given an $N \times N$ matrix A, an $N \times 1$ vector $\vec{b}$ and the equation

$$\begin{bmatrix} A_{11} & A_{12} & \cdots \\ \vdots & \ddots & \\ A_{N1} & & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$$

that is $A\vec{x} = \vec{b}$, solve for $\vec{x}$.

This type of problem is ubiquitous in scientific computing and engineering applications.

Classical methods require at least $O(N)$ operations. $(A^{-1}\vec{b} = \vec{x})$

Harrow Hassidim Lloyd (HHL) Algorithm

Phys. Rev. Lett. 103, 150502 (2009)

Convert vectors to states living in Hilbert space $n = \log_2(N)$ qubits.

$$\vec{b} \to |b\rangle = \frac{\sum_i b_i |i\rangle}{\sum_i |b_i|^2}, \qquad \vec{x} \to |x\rangle = \frac{\sum_i x_i |i\rangle}{\sum_i |x_i|^2}$$

Assume A is Hermitian. If not use instead

$$\tilde{A} = \begin{bmatrix} 0 & A \\ A^\dagger & O \end{bmatrix} = \tilde{A}^\dagger$$

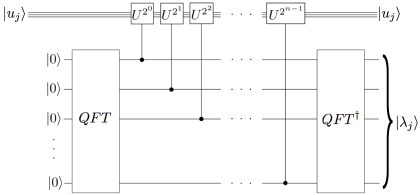
Then the solution is *given by*

$$|x\rangle = \frac{A^{-1} |b\rangle}{\| A^{-1} |b\rangle \|}$$

Let us write $|b\rangle$ in the eigenbasis of A

$$|b\rangle = \sum \beta_j |u_j\rangle \ , \qquad A|u_j\rangle = \lambda_j |u_j\rangle$$

$$A^{-1}|b\rangle = \sum \beta_j \lambda_j^{-1} |u_j\rangle$$



Phase estimation algorithm

Hamiltonian simulation with $U = e^{iAt_0/T}$.

$$|b\rangle |0\rangle_{\text{anc}} \to \sum \beta_j |u_j\rangle |\lambda_j\rangle_{\text{anc}}$$

Next we would like to perform the linear map

$$|\lambda_j\rangle_{\text{anc}} \to C\lambda_j^{-1} |\lambda_j\rangle_{\text{anc}}$$

$|\lambda_j\rangle_{\text{anc}} \rightarrow C\lambda_j^{-1}|\lambda_j\rangle_{\text{anc}}$ is not unitary but can be implemented with finite probability

$$\sum \beta_j |u_j\rangle |\lambda_j\rangle_{\text{anc}} |0\rangle_{\text{a}} \rightarrow \sum \beta_j |u_j\rangle |\lambda_j\rangle_{\text{anc}} \left(\frac{C}{\lambda_j} |1\rangle_{\text{a}} + \sqrt{1 - \frac{C^2}{\lambda_j^2}} |0\rangle_{\text{a}} \right)$$

If the ancillary qubit is measured and found to be in state 1, then we get the state

$$\propto \sum \frac{\beta_j}{\lambda_j} |u_j\rangle |\lambda_j\rangle_{\text{anc}}$$

Finally we undo the phase estimation step to get:

$$|x\rangle \propto \sum \frac{\beta_j}{\lambda_j} |u_j\rangle$$

Let κ : condition number of A
d : number of nonzero entries per row
ϵ : precision with which to prepare $|x\rangle$.

The HHL algorithm takes $poly(\log N, \kappa, 1/\epsilon, d)$ quantum steps to output $|x\rangle$, compared with $poly(N, \kappa, \log(1/\epsilon), d)$ steps required to find $\vec{x}$ using the best known method on a classical computer.

Caveats:

- Finding full answer $\vec{x}$ requires $O(N)$ repetitions to measure the amplitudes of $|x\rangle$.
  - HHL can provide features of $\vec{x}$ such as expectation values over sparse matrices $\vec{x}^\dagger \cdot B \cdot \vec{x}$.
- Input vector $|b\rangle$ needs to be prepared (can dominate complexity)
- The matrix A must be well-conditioned: $\kappa = polylog(N)$ and it must be efficient to simulate e^{iAt}.

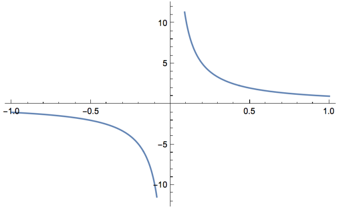
Linear Combination of Unitaries Approach

Childs, Kothari, Somma arXiv:1511.02306

Strategy: Express A^{-1} as a linear combination of easy-to-preform unitaries.

$$A^{-1} \approx \sum_t \alpha_t e^{-iAt}$$

Prepare ancillas in a state proportional to $\sum_t \sqrt{\alpha_t} \, |t\rangle$.
In order to create $|x\rangle \propto A^{-1} |b\rangle$ using LCU requires the ability to implement $C_U = \sum_t |t\rangle \langle t| \otimes e^{-iAt}$ and $Ref(|b\rangle) = \mathbb{1} - 2 |b\rangle \langle b|$.



The function blows up at the origin, but it is sufficient to approximate it in the domain $\left[-1, \frac{1}{\kappa}\right] \cup \left[\frac{1}{\kappa}, 1\right]$

Bottomline: Runtime improved to poly($\log N$, $log(1/\epsilon)$).

An adiabatic algorithm *(with Rolando Somma)*

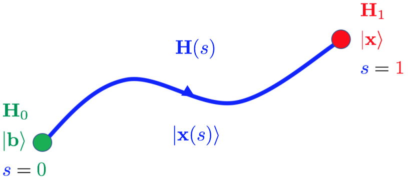
Consider the Hamiltonian

$$H := \frac{|b\rangle\langle b|}{\langle b| A^{-1} |b\rangle} + (\mathbb{1} - A) \, ,$$

$|x\rangle$ is unique eigenstate with largest eigenvalue 1

$$H\left(A^{-1}|b\rangle\right) = \frac{|b\rangle\langle b| A^{-1} |b\rangle}{\langle b| A^{-1} |b\rangle} + (\mathbb{1} - A)A^{-1}|b\rangle = A^{-1}|b\rangle$$

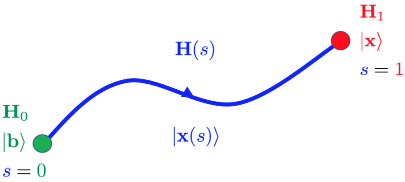
Then, the linear system of equations can be solved by an adiabatic evolution in which s is increased slowly in time or by randomization method.



We define an interpolation such that $H(0) = |b\rangle\langle b|\,,\ H(1) = H$.

$$A(s) = (1 - s)\mathbb{1} + sA$$

$$H(s) = \frac{|b\rangle\langle b|}{\langle b|A(s)^{-1}|b\rangle} + (\mathbb{1} - A(s))$$



It is straightforward to show that

$$|x(s)\rangle = A(s)^{-1}|b\rangle / \|A(s)^{-1}|b\rangle\|$$

is the unique eigenstate of largest eigenvalue 1.

Quantum Zeno Effect

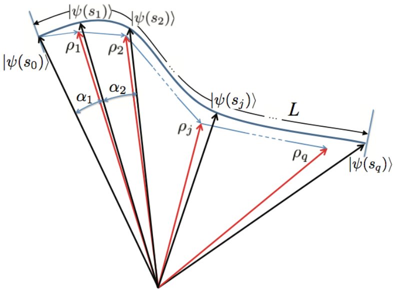
By measuring $H(s)$ for $s_0 = 0 < s_1 < \cdots < s_q = 1$, it is possible to follow the adiabatic path to a given approximation ϵ as long as:

$$|\langle x(s_j)|x(s_{j+1})\rangle|^2 < 1 - \epsilon/q$$

Need to space s_k such that change in $|x(s)\rangle$ (eigenstate of $H(s)$) from s_j to s_{j+1} is uniform and small.
We find that

$$s_j = \frac{1-(1/\kappa)^{j/q}}{1-1/\kappa}$$

with $q = \lceil \log^2(\kappa)/\epsilon \rceil$ is sufficient.



Evolution Randomization Boixo, Knil, Somma (2009)

Evolving with $H(s_k)$ for a random amount of time causes decoherence in the energy eigenbasis:

$$\rho = \sum_{\alpha,\beta} \rho_{\alpha\beta} \left|E_\alpha\right\rangle \left\langle E_\beta\right| \to \sum_{\alpha,\beta} \rho_{\alpha\beta} \int d\mu(t) e^{i(E_\beta - E_\alpha)t} \left|E_\alpha\right\rangle \left\langle E_\beta\right|$$

This effectively simulates a projective measurement if

$$\int d\mu(t) e^{i(E_\beta - E_\alpha)t} \approx \delta_{\alpha,\beta}$$

In fact we only need decoherence for one eigenstate $|x(s)\rangle$ with eigenvalue 1.

$$H(s) = g(s)\,|b\rangle\langle b| + (\mathbb{1} - A(s))\,; \qquad g(s) = \frac{1}{\langle b|\,A(s)^{-1}\,|b\rangle}$$

Assuming we know $g(s)$, sampling t_j uniformly according to

$$t_j \in [0, 1, 2, \dots, Q_j - 1], \quad Q_j = \lceil 2\pi/\Delta(s_j) \rceil$$

is sufficient.

The gap of $H(s)$ satisfies $\Delta(s) \geq 1 - s(1 - 1/\kappa)$.

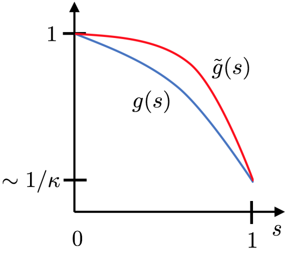
The total runtime of the algorithm $T^* \leq \sum_{j=1}^{q} Q_j \leq 4\pi\kappa \log(\kappa)/\epsilon$.

$$H(s) = g(s)\,|b\rangle\langle b| + (\mathbb{1} - A(s))\,; \qquad g(s) = \frac{1}{\langle b| A(s)^{-1} |b\rangle}$$

The problem is that we don't know $g(s)$. It satisfies:

$$g(0) = 1\,; \qquad g(1) = \langle b| A^{-1} |b\rangle^{-1}$$

First assume that we know $g(1)$ and find $\tilde{g}(s)$ such that:

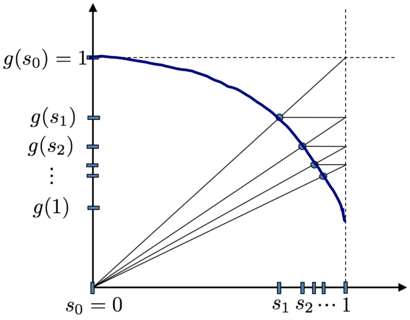


Under these assumptions it can be shown that the Hamiltonian

$$\tilde{H}(s) = \tilde{g}(s)\,|b\rangle\langle b| + (\mathbb{1} - A(s))$$

can be used instead with a total runtime $T^* \leq 16\pi(\kappa+1)\kappa^2/\epsilon$.

Construction of $g(s)$



$g(s)$ is monotonically decreasing.
Given s_{k-1}, $\exists\ s_k > s_{k-1}$ such that

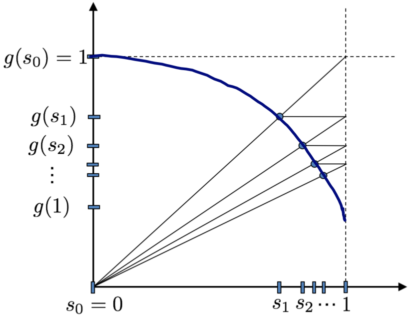
$$g(s_{k-1})\, s_k = g(s_k)$$

$s_0 = 0 < s_1 < \cdots < s_K < \cdots < 1$
converges to 1, s.t $K = O(\kappa \log(\kappa/\epsilon))$
steps are enough.

Define sequence of Hamiltonians

$$H_k := g(s_{k-1})\,|b\rangle\langle b| + (\mathbb{1} - A) = \frac{g(s_{k-1})}{g(s_k)}\left[g(s_k)\,|b\rangle\langle b| + s_k(\mathbb{1} - A)\right] = \frac{g(s_{k-1})}{g(s_k)} H(s_k)$$

Has the same eigenvectors as $H(s_k)$.
Eigenvalue of $|x(s_k)\rangle$ is given by $g(s_{k-1})/g(s_k) = s_k^{-1}$.



$$g(s_{k-1})\, s_k = g(s_k)$$

$$H_k := g(s_{k-1}) \left|b\right\rangle \left\langle b\right| + (\mathbb{1} - A)$$

Assume we know s_{k-1} and $g(s_{k-1})$.

We first prepare $|x(s_k)\rangle$ using H_k.

We then measure the expectation value H_k to obtain

$$s_k = \langle x(s_k)| \, H_k \, |x(s_k)\rangle^{-1}$$

From $g(s_{k-1})$ and s_k we can learn

$$g(s_k) = g(s_{k-1})\, s_k$$

$K = O(\kappa \log(\kappa/\epsilon))$ steps are enough.

Adiabatic Quantum Computation

The evolution induced by the RM is closely related to the coherent evolution induced by the quantum adiabatic method.

assuming $g(s)$ is known ; $T^* \sim \kappa \log(\kappa)/\epsilon$

$$s_j = \frac{1-(1/\kappa)^{j/q}}{1-1/\kappa} \longrightarrow s(t) = \frac{1-(1/\kappa)^{t/T^*}}{1-1/\kappa}$$

assuming $g(1)$ is known ; $T^* \sim \kappa^3/\epsilon$

$$s_j = \frac{\kappa j/q}{1+(\kappa-1)j/q} \longrightarrow s(t) = \frac{\kappa t/T^*}{1+(\kappa-1)t/T^*}$$

assuming nothing ; $O(\kappa \log(\kappa/\epsilon))$ iterations.

While quantum adiabatic approximations that imply these schedules are unknown, they are suggested by the Randomization Method.

Summary

- A certain class of well-conditioned linear system of equations can be solved exponentially faster using quantum computers.
- The HHL algorithm and an algorithm due to Childs, Kothari and Somma runs on universal gate-based quantum computers.
- The adiabatic algorithms require evolutions with (and measurement of expectation values of) Hamiltonians that are linear combinations of A and $|b\rangle\langle b|$.
- The adiabatic algorithm is important in that this problem could be solved using a restricted, maybe non-universal quantum computing device.